Cartesian Decomposition of Linear Continuous Systems

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Reachability problem

We consider systems with linear dynamics, i.e., a trajectory evolves according to ordinary differential equations of the form

$$\dot{x} = Ax$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$.

Given a set of initial states $\mathcal{X}(0)$, we want to compute the set of reachable states $\mathcal{X}(t)$ up to some time bound T. The solution is given analytically as

$$\mathcal{X}(t) = e^{At}\mathcal{X}(0).$$

The computation is exponential in the number of variables.

Approach

We decompose $\mathcal{X}$ into subsystems $\mathcal{Y}$ and $\mathcal{Z}$ as follows. Separate the variables into two blocks

$$x = \begin{pmatrix} y \\ z \end{pmatrix}$$

where $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^l$ with $n = k+l$. This induces a decomposition of A into four blocks:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

The initial states and the dynamics are decomposed to the respective projections of $\mathcal{X}$ to $\mathcal{Y}$ and $\mathcal{Z}$:

$$\mathcal{Y}(0) := \begin{pmatrix} I_k & 0 \end{pmatrix} \mathcal{X}(0)$$
$$\mathcal{Z}(0) := \begin{pmatrix} 0 & I_l \end{pmatrix} \mathcal{X}(0)$$

$$\dot{\mathcal{Y}} := A_{11}\mathcal{Y} + A_{12}\mathcal{Z}(t)$$
$$\dot{\mathcal{Z}} := A_{21}\mathcal{Y}(t) + A_{22}\mathcal{Z}$$

We define our abstraction $\widehat{\mathcal{X}}$ as the Cartesian composition of $\mathcal{Y}$ and $\mathcal{Z}$:

$$\widehat{\mathcal{X}}(t) := \mathcal{Y}(t) \times \mathcal{Z}(t)$$

Example (general case)

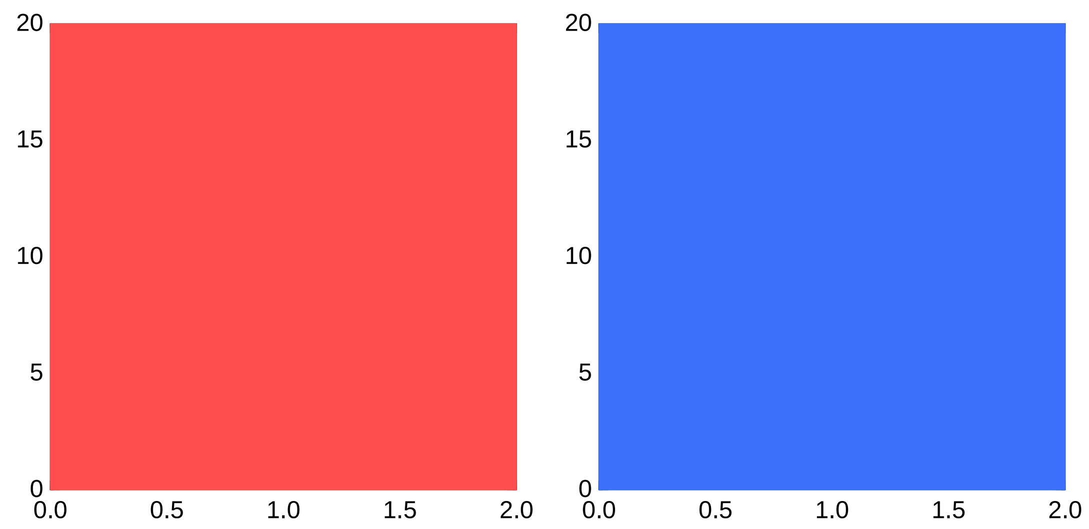
Consider the following system.

$$x = \begin{pmatrix} y \\ z \end{pmatrix}, \quad A = \begin{pmatrix} -12 & 3 \\ 2 & -0.4 \end{pmatrix}$$

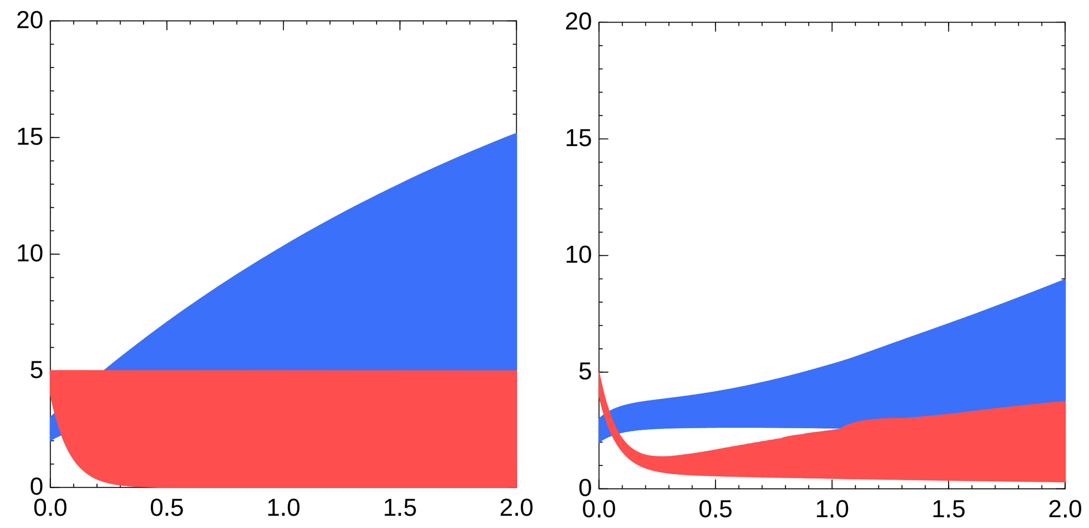
$$4 \leq \mathcal{Y}(0) \leq 5, \quad 2 \leq \mathcal{Z}(0) \leq 3, \quad T = 2$$

with the following initial abstractions:

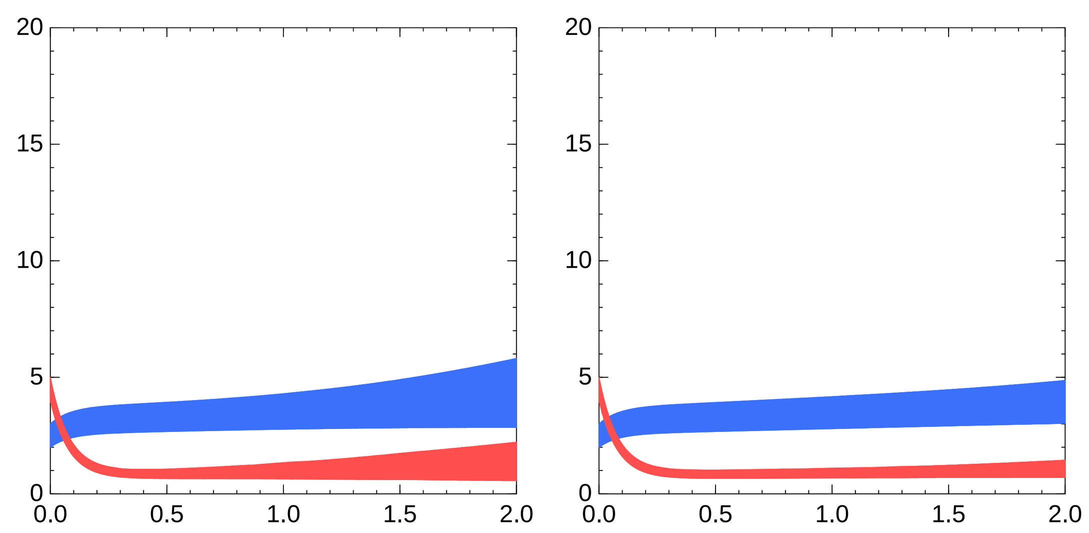
$0 \leq \widehat{\mathcal{Y}}_0(t) \leq 20$ $\quad$ $0 \leq \widehat{\mathcal{Z}}_0(t) \leq 20$



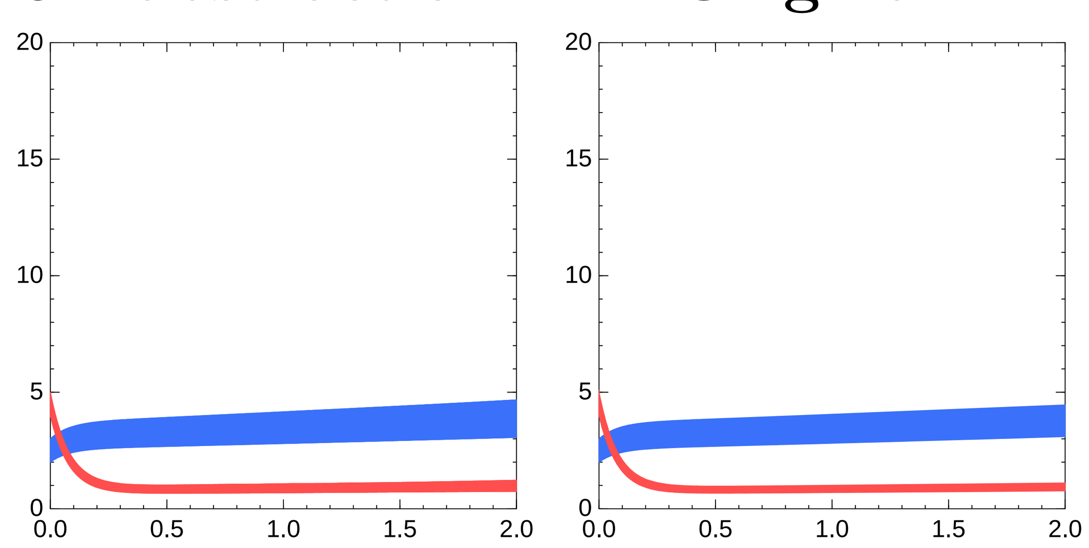
1st abstraction **2nd abstraction**



3rd abstraction **4th abstraction**



5th abstraction **Original**



Approximation

The abstraction overapproximates the original system, i.e.,

$$\widehat{\mathcal{X}}(t) \supseteq \mathcal{X}(t).$$

Equality holds if $\mathcal{X}(0) = \mathcal{Y}(0) \times \mathcal{Z}(0)$.

Different cases

General matrix

Assume initial abstractions $\widehat{\mathcal{Y}}_0(t)$, $\widehat{\mathcal{Z}}_0(t)$. Iteratively refine the abstractions:

$$\dot{\widehat{\mathcal{Y}}}_k = A_{11}\widehat{\mathcal{Y}}_k + A_{12}\widehat{\mathcal{Z}}_{k-1}(t)$$
$$\dot{\widehat{\mathcal{Z}}}_k = A_{21}\widehat{\mathcal{Y}}_{k-1}(t) + A_{22}\widehat{\mathcal{Z}}_k$$

☺ No transformation of initial states

☹ Initial abstraction

☹ Iteration

☹ Errors accumulate

Block-triangular matrix

Assume $A_{21} = 0$.

Subsystem $\mathcal{Z}$ can be analyzed independently. The results need to be propagated to $\mathcal{Y}$.

Every matrix can be transformed to real Schur form with 2×2 sub-blocks B_j by coordinate transformation:

$$A' = \begin{pmatrix} B_1 & \cdots & \cdot \\ 0 & \ddots & \vdots \\ 0 & 0 & B_m \end{pmatrix}$$

☺ Transformation numerically stable

☹ Transformation of initial states

☹ Not parallelizable

☹ Errors accumulate

Block-diagonal matrix

Assume $A_{12} = A_{21} = 0$.

The subsystems are independent.

Every matrix can be transformed to real Jordan form with sub-blocks B_j by coordinate transformation:

$$A' = \begin{pmatrix} B_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & B_m \end{pmatrix}$$

☺ Parallelizable

☺ Errors do not accumulate

☹ Transformation of initial states

☹ Transformation numerically instable