Reachability for weakly nonlinear systems using Carleman linearization

Marcelo Forets, **Christian Schilling**

UNIVERSIDAD DE LA REPÚBLICA URUGUAY
CURE Centro Universitario de la Región Este
AALBORG UNIVERSITET

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Reachability for continuous systems

Carleman linearization

Conservative approximation

Evaluation

Conclusion

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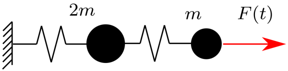
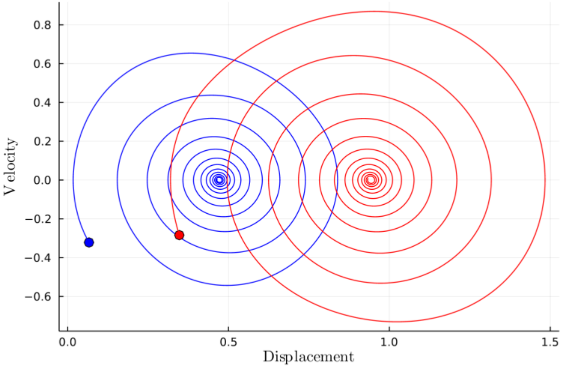
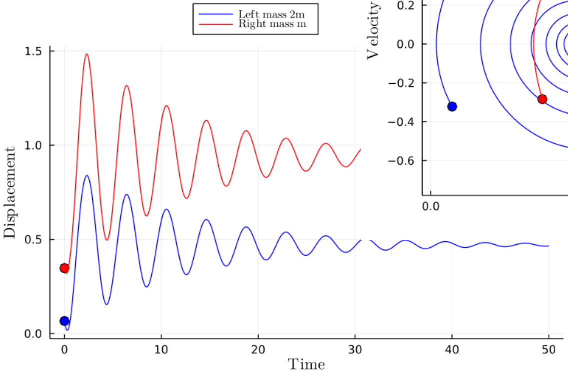
Reachability for linear continuous systems



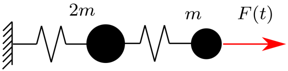


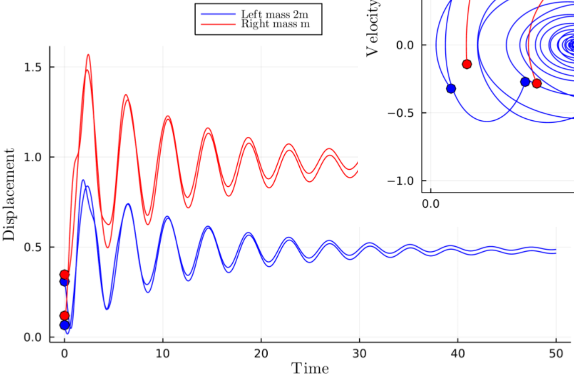


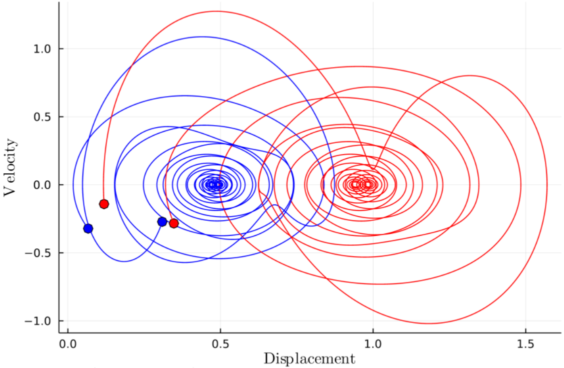
Spring-mass system with two degrees of freedom[1]

[2]Pérez Zerpa, Forets, and Freire Caporale. *Proceedings of the JuliaCon Conferences* (2022).

Reachability for linear continuous systems







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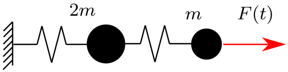

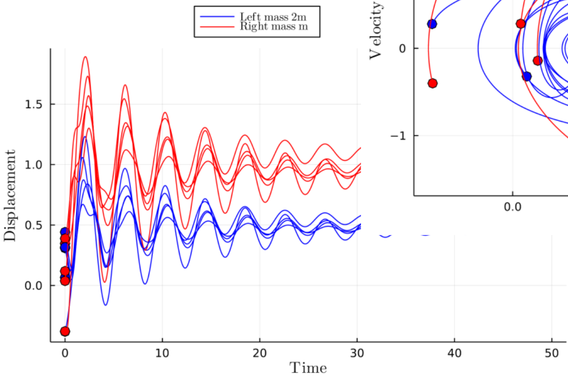

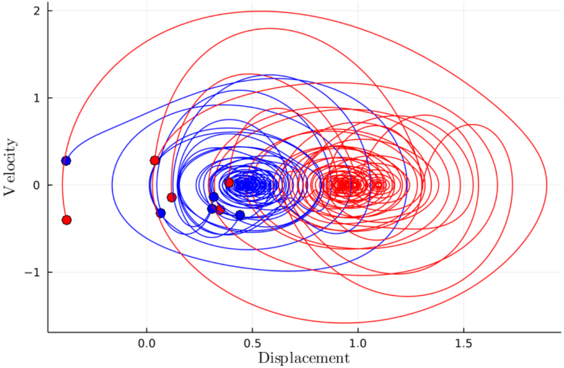

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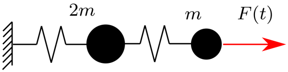

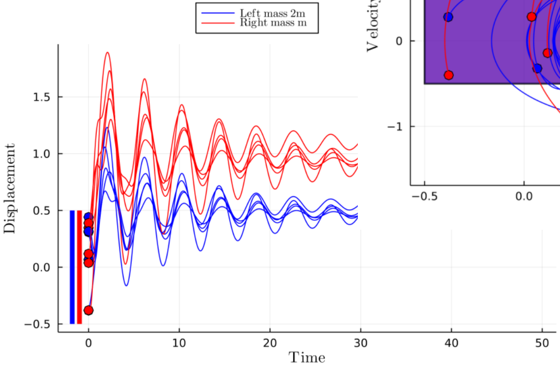

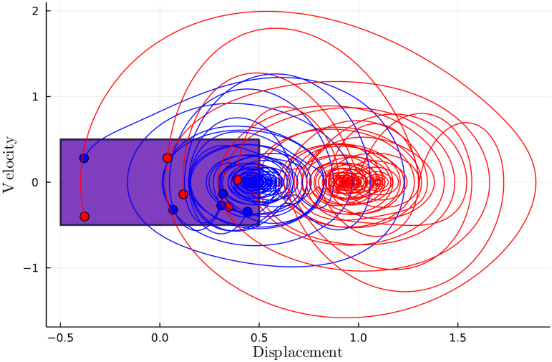

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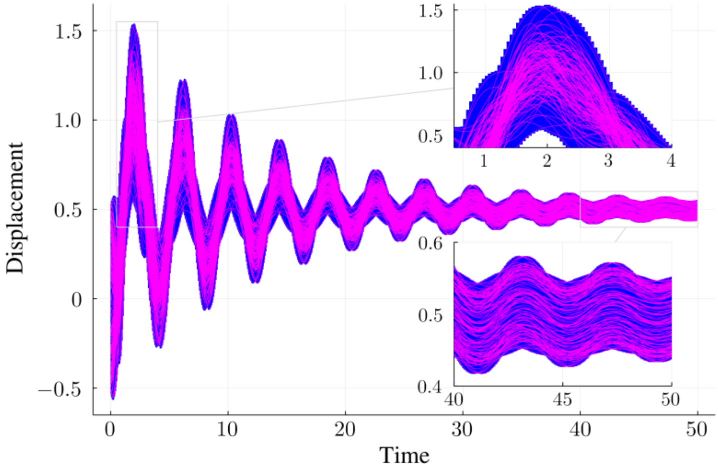
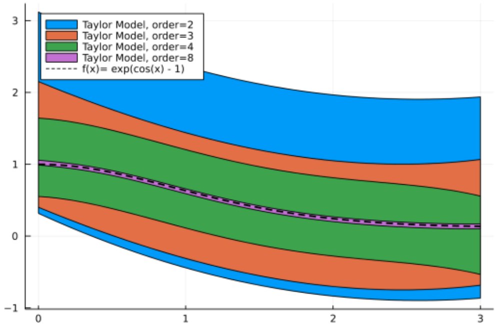


Reachability for nonlinear continuous systems

- Several proposals exist, e.g., based on Taylor models[1]
- $T_3 = 0.394 - 0.393t + 0.182t^2 + 0.014t^3 + [-0.946, 0.803]$
- $T_8 = 0.394 - 0.393t + 0.182t^2 + 0.014t^3 - 0.054t^4 + 0.024t^5 + 0.001t^6 - 0.005t^7 + 0.001t^8 + [-0.041, 0.025]$



[1] Berz and Makino. *Reliab. Comput.* (1998).

State of the art in continuous reachability

- Linear systems
  - Arbitrary precision
  - Wrapping-free algorithms
  - Thousands of dimensions[1]
- Nonlinear systems
  - Arbitrary precision
  - Wrapping effect
  - Only very few dimensions

[1] Bogomolov, Forets, Frehse, Podelski, and Schilling. *Inf. Comput.* (2022).

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Kronecker product

$$x \otimes x := (x_1^2, x_1 x_2, x_2 x_1, x_2^2)^T \quad (x \in \mathbb{R}^2)$$

$$x^{\otimes k} := \underbrace{x \otimes \cdots \otimes x}_{k \text{ times}}$$

$$A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{pmatrix}$$

Quadratic ODEs

- Polynomial ODEs can be reduced to quadratic form

$$\frac{dx(t)}{dt} = \underbrace{F_1 x}_{\text{"linear behavior"}} + \underbrace{F_2 x^{\otimes 2}}_{\text{"nonlinear behavior"}} \quad (1)$$

- Assume that F_1 and F_2 are time invariant

Carleman linearization[1]

- Assume a quadratic system (1) $\frac{dx(t)}{dt} = F_1 x + F_2 x^{\otimes 2}$ of dimension n with initial condition $x(0) = x_0$
- Introducing auxiliary variables $\hat{y}_j := x^{\otimes j}, j > 0$ leads to equivalent but infinite linear system
- Truncation at order N yields approximation

$$\frac{d\hat{y}(t)}{dt} = A\hat{y} \tag{2}$$

  where $\hat{y}(0) = \hat{y}_0 = (x_0, x_0^{\otimes 2}, \dots, x_0^{\otimes N})^T$ and A on next slide
- Dimension of (2) is $\mathcal{O}(n^N)$

[1] Carleman. *Acta Mathematica* (1932).

Carleman linearization[1]

$$A := \begin{pmatrix} A_1^1 & A_2^1 & 0 & 0 & \cdots & 0 \\ 0 & A_2^2 & A_3^2 & 0 & \cdots & 0 \\ 0 & 0 & A_3^3 & A_4^3 & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & A_{N-1}^{N-1} & A_N^{N-1} \\ 0 & 0 & \cdots & 0 & 0 & A_N^N \end{pmatrix}$$

$$A_{i+i'-1}^i := \sum_{\nu=1}^{i} \overbrace{\mathbb{I}_n \otimes \cdots \otimes \underset{\substack{\uparrow \\ \nu\text{-th position}}}{F_{i'}} \otimes \cdots \otimes \mathbb{I}_n}^{\text{i factors}} \quad (i' \in \{1, 2\})$$

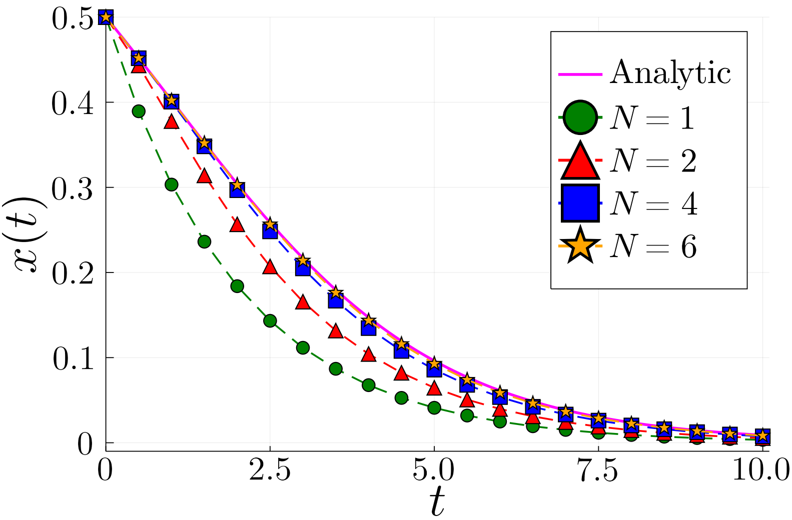
[1] Carleman. *Acta Mathematica* (1932).

Example: Logistic equation

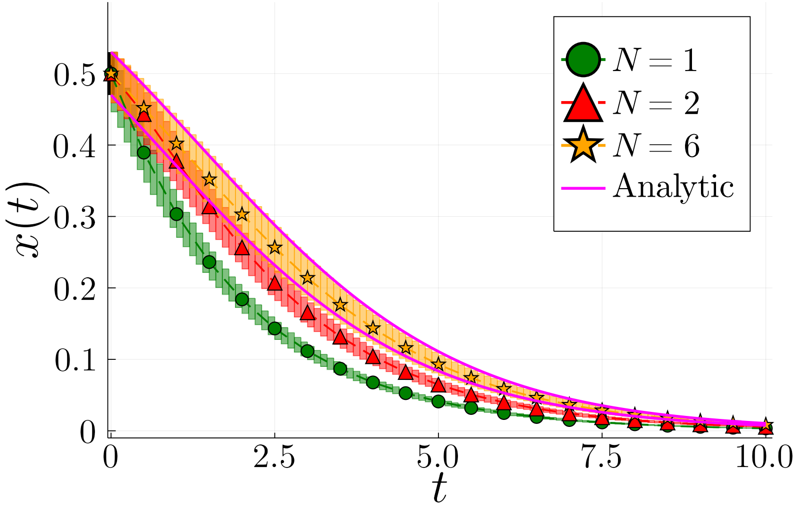
- $\dfrac{dx(t)}{dt} = rx\left(1 - \dfrac{x}{K}\right) \quad r > 1, K > 0$
- Quadratic form: $\dfrac{dx(t)}{dt} = ax + bx^2$ where $a = r, b = -\dfrac{r}{K}$
- Lifting: $\hat{y}_j := x^j$ with derivatives $\hat{y}_j' = ja\hat{y}_j + jb\hat{y}_{j+1} \quad (j > 0)$
- Truncate at order $N = 4$:

$$\frac{d\hat{y}(t)}{dt} = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 2a & 2b & 0 \\ 0 & 0 & 3a & 3b \\ 0 & 0 & 0 & 4a \end{pmatrix} \hat{y}, \quad \hat{y}(0) = \begin{pmatrix} x_0 \\ x_0^2 \\ x_0^3 \\ x_0^4 \end{pmatrix}$$

Example: Logistic equation



Example: Logistic equation



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Error bound

$$\frac{dx(t)}{dt} = F_1 x + F_2 x^{\otimes 2} \tag{1}$$

- Let λ_1 be the eigenvalue of F_1 with largest real part
- We call (1) **weakly nonlinear** if $R := \frac{\|x_0\|\|F_2\|}{|\mathrm{Re}(\lambda_1)|} < 1$
- We call (1) **dissipative** if $\mathrm{Re}(\lambda_1) < 0$
- Error of j-th block of variables is $\eta_j(t) := x^{\otimes j}(t) - \hat{y}_j(t)$

Theorem[1]

If (1) is weakly nonlinear and dissipative, the error of the N-truncated linear system satisfies (for all $t \geq 0$)

$$\|\eta_j(t)\| \leq \|x_0\| R^N (1 - e^{\mathrm{Re}(\lambda_1)t})^N$$

[1] Liu et al. *Proc. Natl. Acad. Sci.* (2021).

Overview

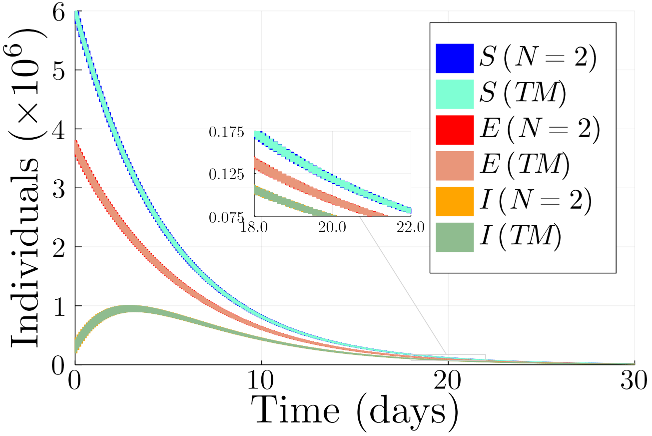
Evaluation: SEIR model[1]

- $F_1 = \begin{pmatrix} -\frac{\Lambda}{P} - r_{\text{vac}} & 0 & 0 \\ 0 & -\frac{\Lambda}{P} - \frac{1}{T_{\text{lat}}} & 0 \\ 0 & \frac{1}{T_{\text{lat}}} & -\frac{\Lambda}{P} - \frac{1}{T_{\text{inf}}} \end{pmatrix}$
- $F_2 = \begin{pmatrix} 0 & 0 & -\frac{r_{\text{tra}}}{P} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{r_{\text{tra}}}{P} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$
- $R \approx 0.68$, $\mathrm{Re}(\lambda_1) \approx -0.19$

[1] Pan et al. *JAMA* (2020).

Evaluation: SEIR model[1]

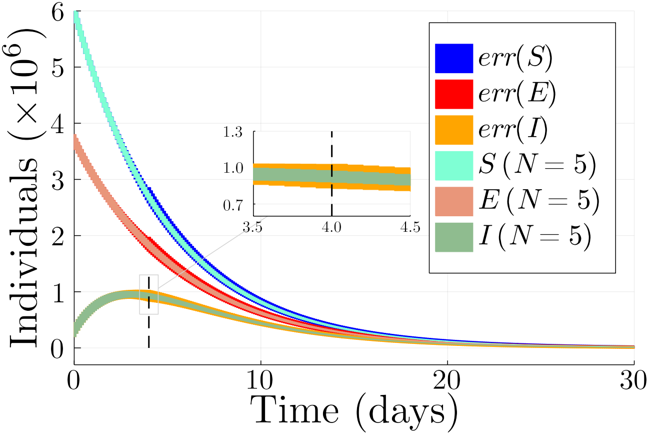
- No error estimation



[1] Pan et al. *JAMA* (2020).

Evaluation: SEIR model[1]

- Error estimation and re-estimation at $t = 4$



[1] Pan et al. *JAMA* (2020).

Evaluation: SEIR model[1]

	no error bound	incl. error bound
TM	6.14 s	
Carleman	$N = 2$: 0.006 s	$N = 5$: 0.185 s

[1] Pan et al. *JAMA* (2020).

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Conclusion and future work

- Carleman linearization of quadratic systems
- Reachability analysis for set-based approximation
  - Weakly nonlinear and dissipative systems
  - Low orders often suffice
  - Can be faster than nonlinear solvers
  - Error bound for conservative results (wrapping-free!)

Future work

- Exploit problem structure (Kronecker product, sparse block-bidiagonal matrix, . . .)
- Automatic re-estimation of error bounds
- Initial condition beyond hyperrectangles