Calculus with Convex Sets

in a Nutshell

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April 2, 2019

Overview

Overview

Convex sets

- We consider the vector space $\mathbb{R}^n$

Definition (Convex set)

X is convex if $X = \{\lambda \cdot \vec{x} + (1 - \lambda) \cdot \vec{y} \mid \vec{x}, \vec{y} \in X, \lambda \in [0, 1] \subseteq \mathbb{R}\}$

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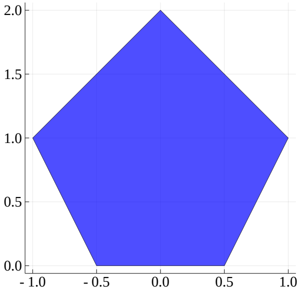
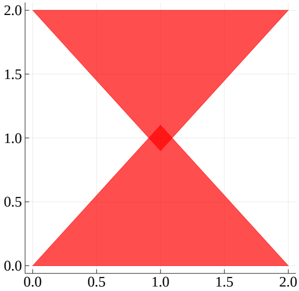


Convex

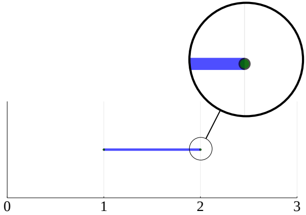


Not convex

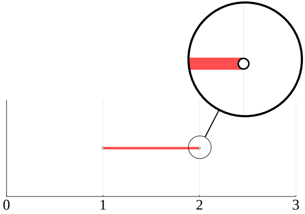
Compact sets

Definition (Closed set)

A set is *closed* if it contains all its boundary points



$1 \leq x_1 \leq 2 \quad (\subseteq \mathbb{R}^1)$



$1 < x_1 < 2 \quad (\subseteq \mathbb{R}^1)$

Compact sets

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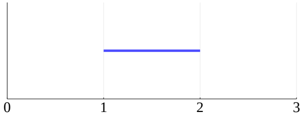
Definition (Bounded set)

X is *bounded* if $\quad \exists \delta \in \mathbb{R}\ \forall \vec{x}, \vec{y} \in X : \|x - y\| \leq \delta$



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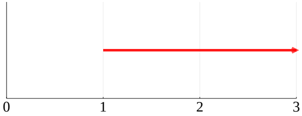

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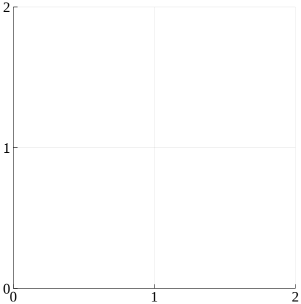
Definition (Compact set)

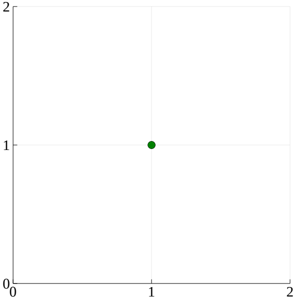
A set is *compact* if it is closed and bounded

Overview

Simplest examples

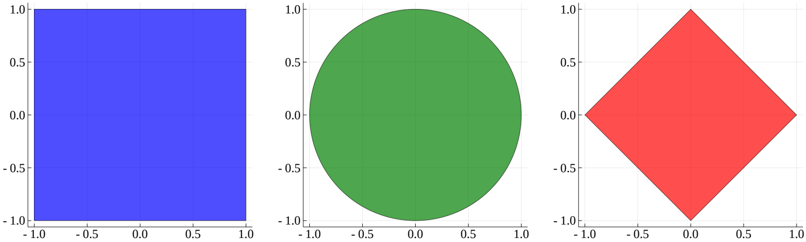


Empty set

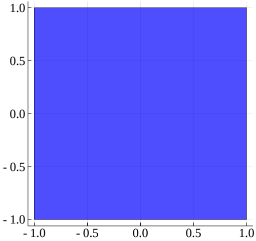


Singleton

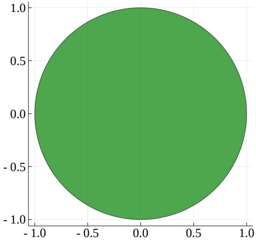
More examples



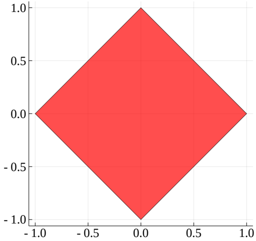
Unit balls



Unit ball in ∞-norm
aka hypercube



Unit ball in 2-norm
aka hypersphere



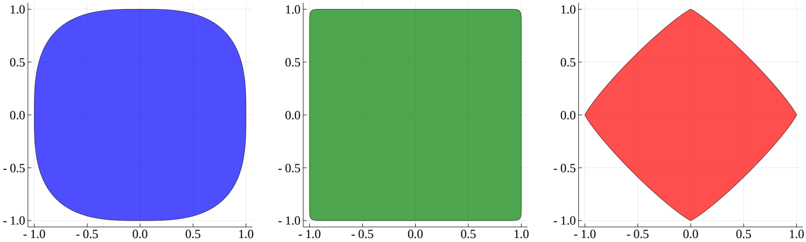
Unit ball in 1-norm
aka cross-polytope

Definition (p-norm)

$$\|\vec{x} = (x_1, \ldots, x_n)\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}.$$

- Balls in the p-norm are convex for $p \geq 1$.

Unit balls



Unit ball in 3-norm | Unit ball in 42-norm | Unit ball in $(\pi - 2)$-norm

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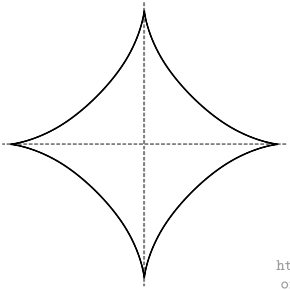
Unit balls



https://commons.wikimedia.org/wiki/File:Astroid.svg

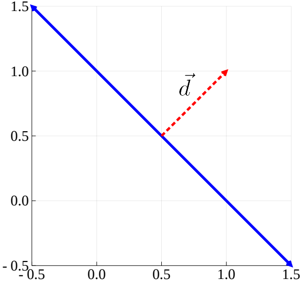
Unit ball in 2/3-norm (not convex!)

Definition (p-norm)

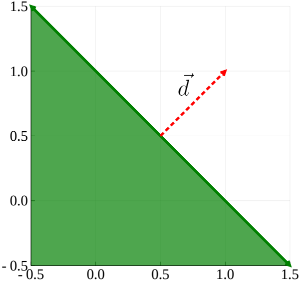
$$\|\vec{x} = (x_1, \ldots, x_n)\|_p := \sqrt[p]{|x_1|^p + \cdots + |x_n|^p}.$$

- Balls in the p-norm are convex for $p \geq 1$.

Unbounded sets



Hyperplane $\langle \vec{d}, \vec{x} \rangle = c$



Half-space $\langle \vec{d}, \vec{x} \rangle \leq c$

Overview

Preliminaries

Basic convex sets

Set operations

Advanced convex sets

Support function

Conclusion

Minkowski sum

Definition

$$X \oplus Y := \{\vec{x} + \vec{y} \mid \vec{x} \in X, \vec{y} \in Y\}$$

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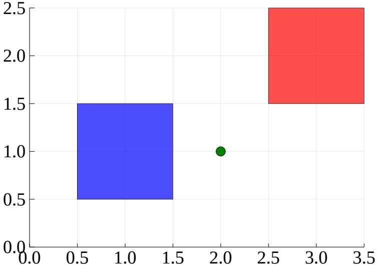
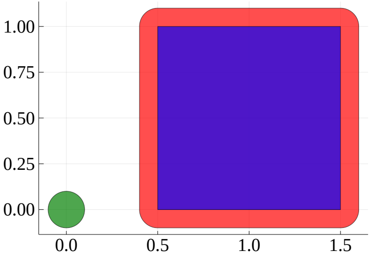


Translation

Minkowski sum

Definition

$$X \oplus Y := \{\vec{x} + \vec{y} \mid \vec{x} \in X, \vec{y} \in Y\}$$



Square $\oplus$ circle centered in the origin

Linear map

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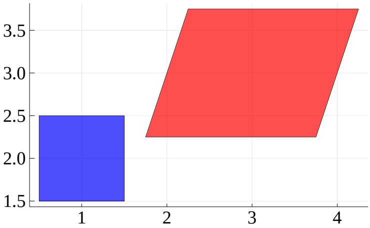
$$M = \begin{pmatrix} 2 & 0.5 \\ 0 & 1.5 \end{pmatrix}$$



Invertible map

Linear map

Definition

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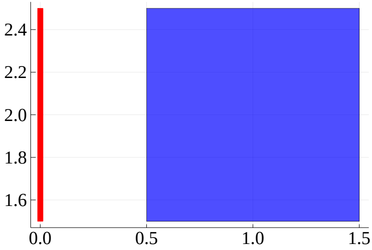
$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$



Projection

Convex hull

Definition

$CH(X) :=$ smallest set Y s.t.

$$Y = X \cup \{\lambda \cdot \vec{x} + (1 - \lambda) \cdot \vec{y} \mid \vec{x}, \vec{y} \in Y, \lambda \in [0, 1] \subseteq \mathbb{R}\}$$

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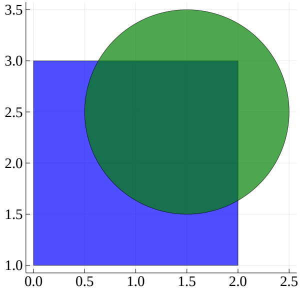

Union (not convex)

Convex hull

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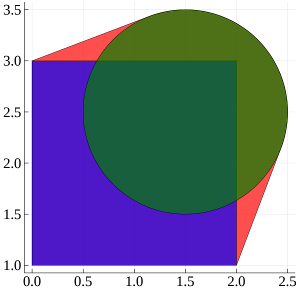

Convex hull of the union

Intersection

Definition

$$X \cap Y := \{\vec{x} \mid \vec{x} \in X \wedge \vec{x} \in Y\}$$

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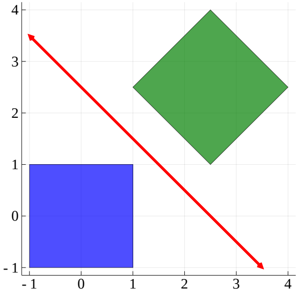

Disjoint

Intersection

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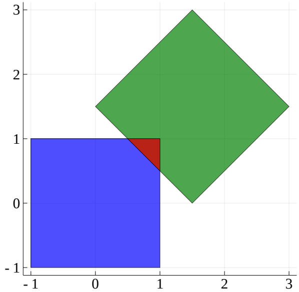
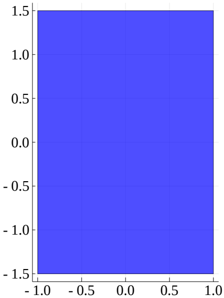


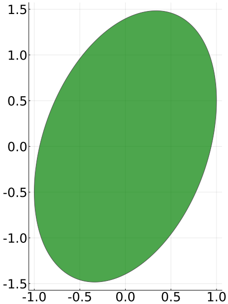
Intersecting

Overview

Generalizations of balls



Hyperrectangle

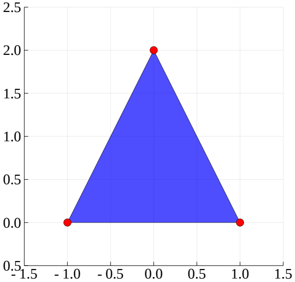


(Hyper-)Ellipsoid

Polytopes

Definition (Vertex representation)

A *polytope* is the convex hull of the union of finitely many singletons



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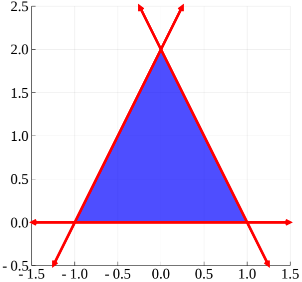
Definition (Constraint (or half-space) representation)

A *polytope* is the bounded intersection of finitely many half-spaces



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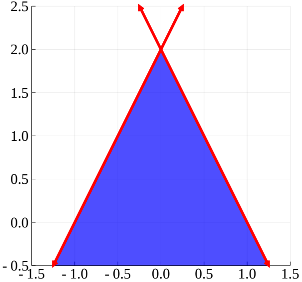

Polyhedron (unbounded)

Zonotopes

- Minkowski sum of line segments

$$\left\{ \vec{c} + \sum_{i=1}^{p} \xi_i \cdot \vec{g}_i \;\middle|\; \xi_i \in [-1, 1] \right\}$$

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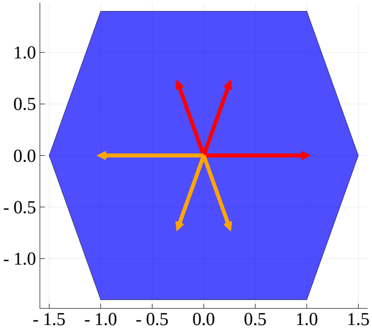

- Centrally symmetric polytope

Closure properties

	$X \oplus Y$	$M \cdot X$	$CH(X \cup Y)$	$X \cap Y$
Hyperrectangle	☺	☹	☹	☺[1]
Ellipsoid	☹	☺	☹	☹
Zonotope	☺	☺	☹	☹
Polytope	☺	☺	☺	☺

[1] Unless the intersection is empty

Overview

Support function

Definition (Support function)

Let $\emptyset \subsetneq X \subseteq \mathbb{R}^n$ be a compact convex set and $\vec{d} \in \mathbb{R}^n$ a direction

$$\rho_X : \mathbb{R}^n \to \mathbb{R}$$
$$\rho_X(\vec{d}) := \max_{\vec{x} \in X} \langle \vec{d}, \vec{x} \rangle$$

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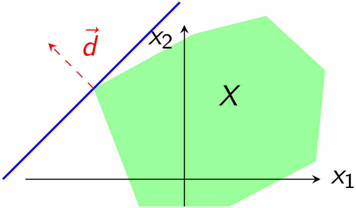

$$\{\vec{x} \in \mathbb{R}^2 : \langle \vec{d}, \vec{x} \rangle \leq \rho_X(\vec{d})\}$$

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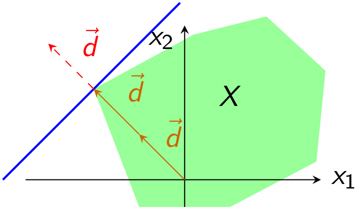


$\{\vec{x} \in \mathbb{R}^2 : \langle \vec{d}, \vec{x} \rangle \leq \rho_X(\vec{d})\}$

Properties of the support function

Proposition

- $\rho_{\lambda \cdot X}(\vec{d}) = \rho_X(\lambda \cdot \vec{d})$
- $\rho_{X \oplus Y}(\vec{d}) = \rho_X(\vec{d}) + \rho_Y(\vec{d})$
- $\rho_{M \cdot X}(\vec{d}) = \rho_X(M^T \cdot \vec{d})$
- $\rho_{CH(X \cup Y)}(\vec{d}) = \max(\rho_X(\vec{d}), \rho_Y(\vec{d}))$

Proposition

For every compact convex set $X \neq \emptyset$ and $D \subseteq \mathbb{R}^n$ we have

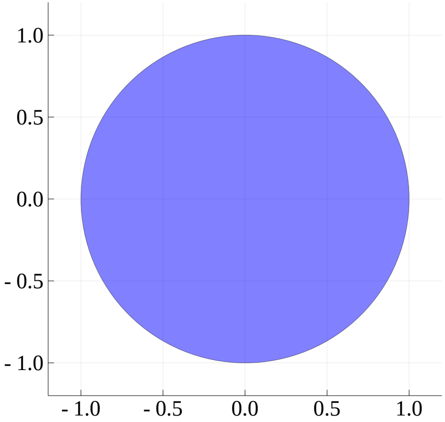
$$X \subseteq \bigcap_{\vec{d} \in D} \langle \vec{d}, \vec{x} \rangle \leq \rho_X(\vec{d})$$

and equality holds for $D = \mathbb{R}^n$

Complexity

- Optimization of a linear function over a convex set
  $\rightarrow$ convex optimization (efficient!)
- Even more efficient for specific set representations
  - Polytopes: linear program
  - Zonotopes: $\mathcal{O}(p \cdot n^2)$ (p generators)
  - Ellipsoids: $\mathcal{O}(n^2)$
  - Hyperrectangles: $\mathcal{O}(n)$
- Let $c(X)$ be the complexity for set X
  - $X \oplus Y$: $\mathcal{O}(c(X) + c(Y))$
  - $M \cdot X$: $\mathcal{O}(n^2 + c(X))$
  - $CH(X, Y)$: $\mathcal{O}(c(X) + c(Y))$

Overapproximation using support function



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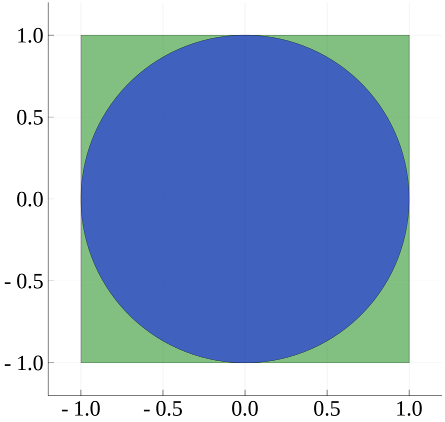

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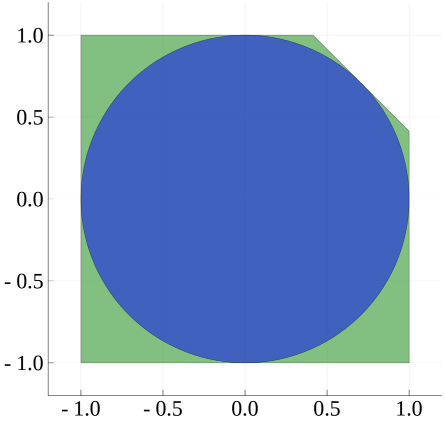

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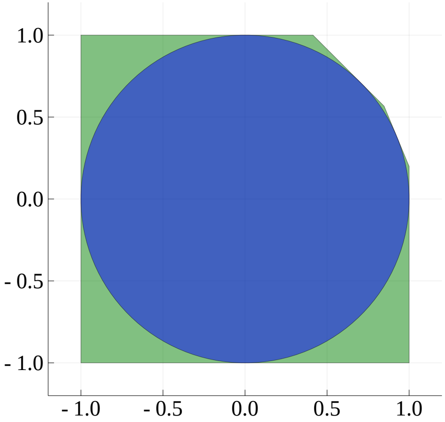

- Template directions
- ε-close approximation

Overview

Conclusion

- Convex sets are expressive
- Closure under most standard set operations
- Support function allows for efficient lazy computations
- Non-convex sets: approximate by (union of) convex sets

- Implemented in the Julia package `LazySets` (joint work with Marcelo Forets from Universidad de la República, Uruguay) and available on Github