Conservative Time Discretization: A Comparative Study

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Reachability

- Given an n-dimensional **linear** **continuous** system

$$\dot{x}(t) = Ax(t)$$

and a set of initial states $x(0) \in \mathcal{X}_0 \subseteq \mathbb{R}^n$

- Let $\xi_{x_0}(t)$ be the trajectory from initial state x_0 at time point t
- We are interested in the **reachable states** for time point t

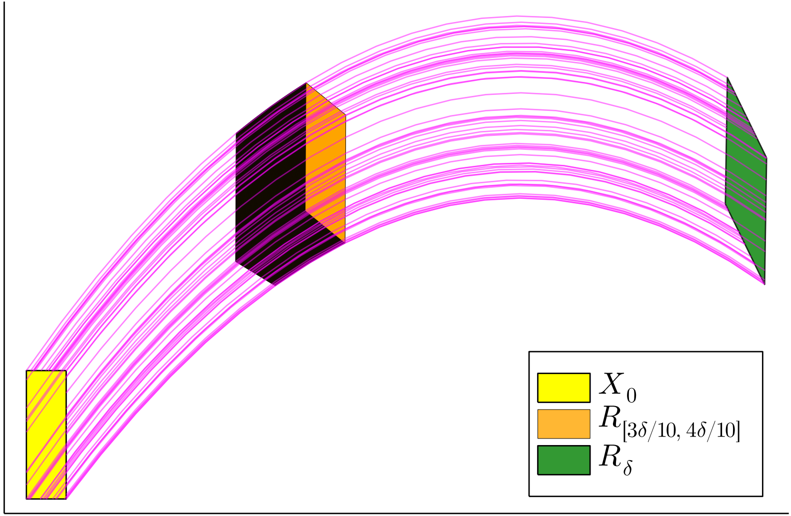
$$\mathcal{R}_t = \{\xi_{x_0}(t) : x_0 \in \mathcal{X}_0\}$$

and more generally for time intervals

$$\mathcal{R}_{[t_0,t_1]} = \{\xi_{x_0}(t) : x_0 \in \mathcal{X}_0, t \in [t_0, t_1]\}$$

- **Time-bounded reachability problem**: Compute $\mathcal{R}_{[0,T]}$

Reachability



Reachability

- Given an n-dimensional **linear** **continuous** system

$$\dot{x}(t) = Ax(t)$$

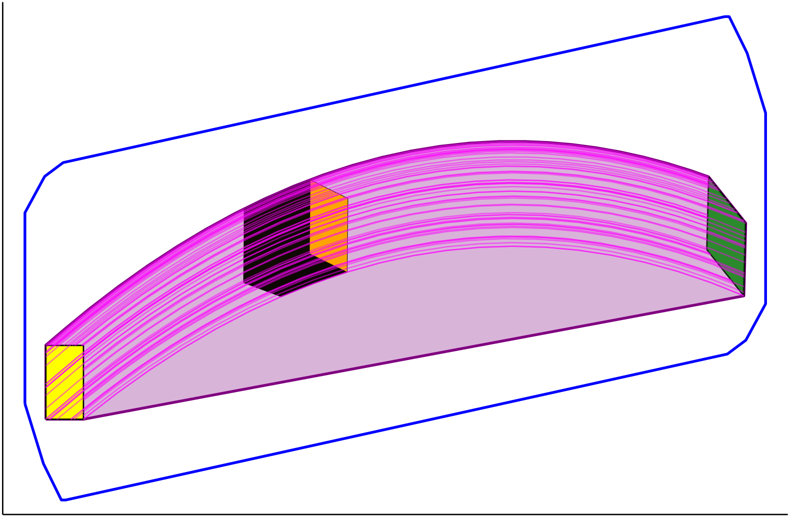
and a set of initial states $x(0) \in \mathcal{X}_0 \subseteq \mathbb{R}^n$

- General idea: Exploit that for any $t_0, t_1, \delta \in \mathbb{R}_{\geq 0}$ and $\Phi := e^{A\delta}$

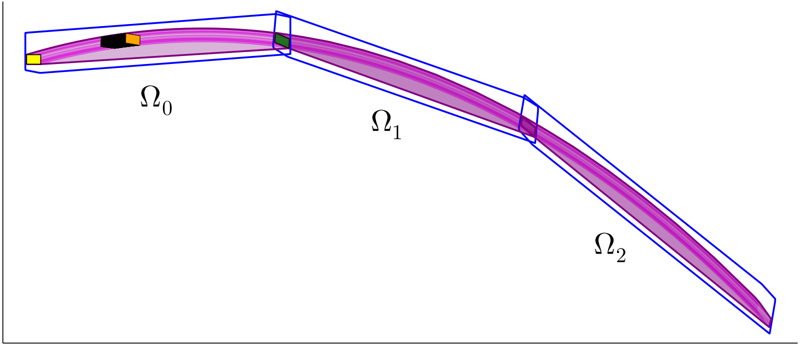
$$\mathcal{R}_{[t_0+\delta, t_1+\delta]} = \Phi \mathcal{R}_{[t_0, t_1]}$$

and compute (over)approximation $\Omega_0 \supseteq \mathcal{R}_{[0,\delta]}$

Discretization



Discretization



- Define the sequence $\Omega_{k+1} = \Phi\Omega_k$
- Simple corollary:

$$\Omega_0 \supseteq \mathcal{R}_{[0,\delta]} \implies \bigcup_{k=0}^{\lceil T/\delta \rceil} \Omega_k \supseteq \mathcal{R}_{[0,T]}$$

Discretization

- Given an n-dimensional **linear** **continuous** system

$$\dot{x}(t) = Ax(t)$$

and a set of initial states $x(0) \in \mathcal{X}_0 \subseteq \mathbb{R}^n$

- **Discretization algorithm**
  - Choose (small) time step $\delta \in \mathbb{R}_{>0}$
  - Compute (over)approximation $\Omega_0 \supseteq \mathcal{R}_{[0,\delta]}$
  - Transform to n-dimensional **linear** **discrete** system

$$x_{k+1} = \Phi x_k$$

with initial states $x_0 \in \Omega_0$

Discretization

- Central problem: "Compute approximation $\Omega_0 \supseteq \mathcal{R}_{[0,\delta]}$"
- **Reachability algorithm** for discretized system
  - **Precision** depends **only** on Ω_0
  - For performance reasons, Ω_0 should be **convex**
    (And sometimes more specific, e.g., a **zonotope**)
- Secondary problem: How to **choose** δ?
  - Large: fast (few iterations for **reachability algorithm**)
  - Small: precise (roughly: $\lim_{\delta \to 0} \Omega_0 \to \mathcal{R}_{[0,\delta]}$)
  - Not covered here
- Implemented in **JuliaReach**[1]

[1] S. Bogomolov et al. *HSCC*. `https://github.com/JuliaReach/`. 2019.

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Inputs

- Most approaches support systems of the form

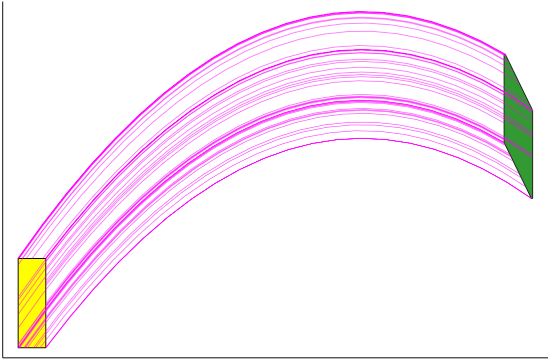
$$\dot{x}(t) = Ax(t) + Bu(t)$$

  where $u(t)$ is an **input signal** coming from a known set $\mathcal{U}$: $u(t) \in \mathcal{U}$ for all t
- For simplicity we only consider **homogeneous systems** ($\mathcal{U} = \{\mathbf{0}\}$) in this presentation

Notation

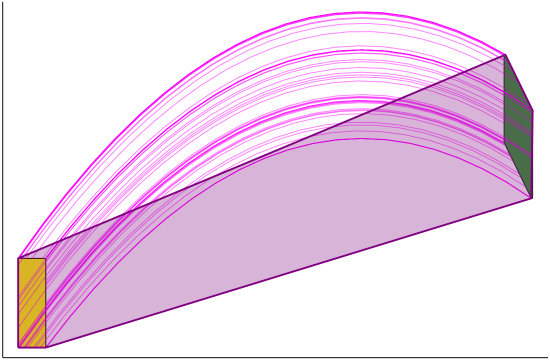
- $CH(\mathcal{X})$: **convex hull** of $\mathcal{X}$
  (smallest convex set containing $\mathcal{X}$)
- $\mathcal{X} \oplus \mathcal{Y} = \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$: **Minkowski sum** of $\mathcal{X}$ and $\mathcal{Y}$
- $\mathcal{B}_\varepsilon^p$: **ball** in p-norm of radius ε centered in origin (may omit p)
- $\boxdot(\mathcal{X})$: **symmetric interval hull** of $\mathcal{X}$
  (smallest box containing both $\mathcal{X}$ and its reflection in origin)

Generic approach



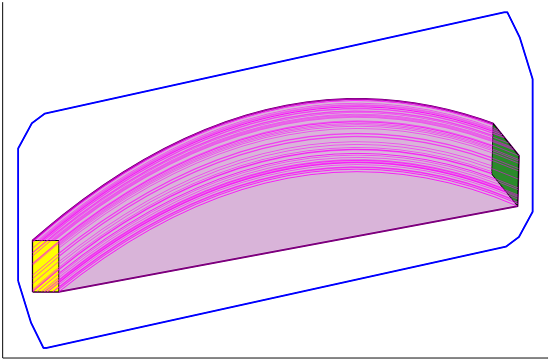
- Start with $\mathcal{X}_0$ and $\Phi\mathcal{X}_0$

Generic approach



- Compute convex hull $CH(\mathcal{X}_0 \cup \Phi\mathcal{X}_0)$

Generic approach



- Bloat the set to cover all trajectories

$$CH(\mathcal{X}_0 \cup (\Phi\mathcal{X}_0 \oplus \mathcal{H})) \oplus \mathcal{J}$$

First- and second-order methods

- **d/dt**[1]:

$$\Omega_0 = CH(\mathcal{X}_0 \cup \Phi\mathcal{X}_0) \oplus \mathcal{B}_\varepsilon$$
$$\varepsilon = \left(e^{\|A\|\delta} - 1 - \|A\|\delta\right)\|\mathcal{X}_0\| - \frac{3}{8}\|A\|^2\delta^2\|\mathcal{X}_0\|$$

- **Zonotope**[2]:

$$\Omega_0 = zonotope(CH(\mathcal{X}_0 \cup \Phi\mathcal{X}_0)) \oplus \mathcal{B}_\varepsilon^\infty$$
$$\varepsilon = \left(e^{\|A\|_\infty\delta} - 1 - \|A\|_\infty\delta\right)\|\mathcal{X}_0\|_\infty$$

- **LGG**[3]:

$$\Omega_0 = CH(\mathcal{X}_0 \cup (\Phi\mathcal{X}_0 \oplus \mathcal{B}_\varepsilon))$$
$$\varepsilon = \left(e^{\|A\|\delta} - 1 - \|A\|\delta\right)\|\mathcal{X}_0\|$$

[1] E. Asarin et al. *HSCC.* 2000

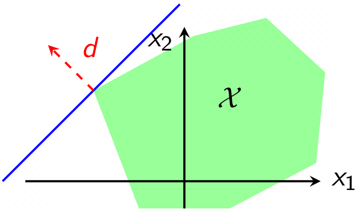
[2] A. Girard. *HSCC.* 2005

[3] C. Le Guernic and A. Girard. *Nonlinear Analysis: Hybrid Systems* (2010)

Support function

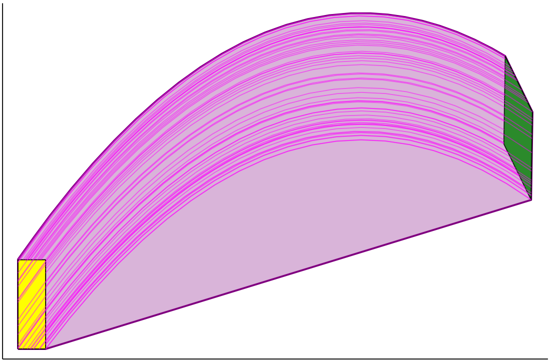
- Let $\emptyset \subsetneq \mathcal{X} \subseteq \mathbb{R}^n$ be a **compact convex set** and $d \in \mathbb{R}^n$
  The **support function** of $\mathcal{X}$ in direction d is

$$\rho_{\mathcal{X}} : \mathbb{R}^n \to \mathbb{R}$$
$$\rho_{\mathcal{X}}(d) = \max_{x \in \mathcal{X}} \langle d, x \rangle$$



$\{x \in \mathbb{R}^n : \langle d, x \rangle \leq \rho_{\mathcal{X}}(d)\}$

Best convex approximation



Forward and forward-backward methods

- **Forward-backward (SpaceEx)**[1]:

$$\Omega_0 = CH\Big(\bigcup_{\lambda\in[0,1]} \mathcal{Y}_\lambda\Big)$$

$$\mathcal{Y}_\lambda = (1-\lambda)\mathcal{X}_0 \oplus \lambda\Phi\mathcal{X}_0 \oplus (\lambda E_+ \cap (1-\lambda)E_-)$$

$$E_+ = \boxdot(\Psi(|A|,\delta)\boxdot(A^2\mathcal{X}_0))$$

$$E_- = \boxdot(\Psi(|A|,\delta)\boxdot(A^2\Phi\mathcal{X}_0))$$

$$\Psi(A,\delta) = \sum_{i=0}^{\infty}\frac{\delta^{i+2}}{(i+2)!}A^i$$

- **Forward (JuliaReach)**[2]:

$$\Omega_0 = CH(\mathcal{X}_0 \cup (\Phi\mathcal{X}_0 \oplus E_+))$$

[1] G. Frehse et al. *CAV*. 2011

[2] S. Bogomolov et al. *HSCC*. 2018

Correction-hull method

- **Interval matrices (CORA)**[1]:

$$\Omega_0 = CH(\mathcal{X}_0 \cup \Phi \mathcal{X}_0) \oplus F_p \mathcal{X}_0$$

$$F_p = E + \sum_{i=2}^{p} [\delta^i (i^{\frac{-i}{i-1}} - i^{\frac{-1}{i-1}}), 0] \frac{A^i}{i!}$$

$$E = n \times n \text{ matrix filled with } [-\varepsilon, \varepsilon]$$

$$\varepsilon = \frac{(\|A\|_\infty \delta)^{p+1}}{(p+1)!} \frac{1}{1-\alpha}$$

$$\alpha = \frac{\|A\|_\infty \delta}{p+2} \overset{!}{<} 1$$

- Truncation order $p = 4$ used in experiments

[1] M. Althoff, O. Stursberg, and M. Buss. *CDC*. 2007.

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Experiment 1

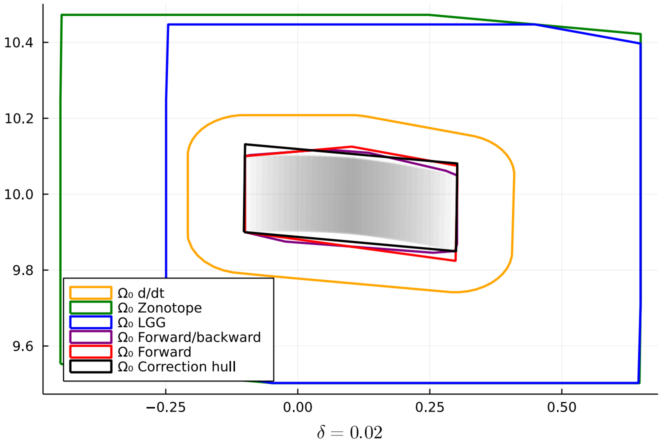
- **Harmonic oscillator**

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ -4\pi & 0 \end{pmatrix} x(t)$$
$$x_0 = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$$

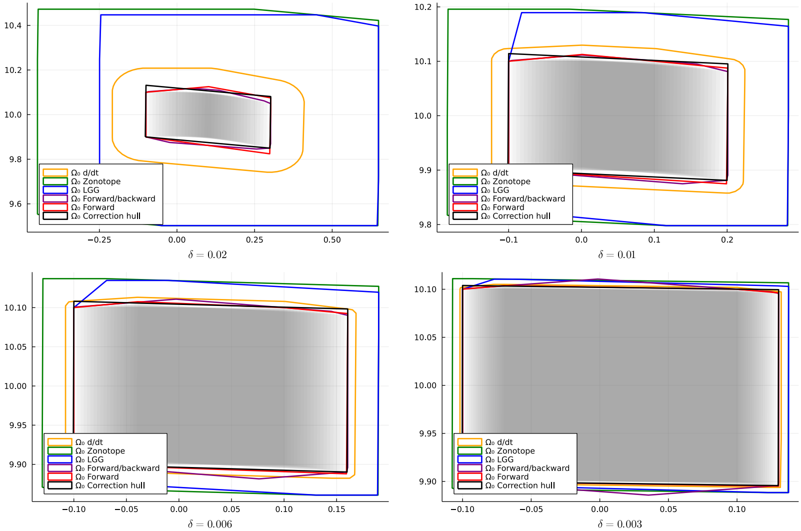
- Compare methods
- **Vary one parameter** (δ resp. $\mathcal{X}_0$)
- **Reference reachable states** for small time steps in gray

Experiment 1 - Example 1

- $\mathcal{X}_0 = [-0.1, 0.1] \times [9.9, 10.1] = \square(x_0, 0.1)$ (square around x_0)

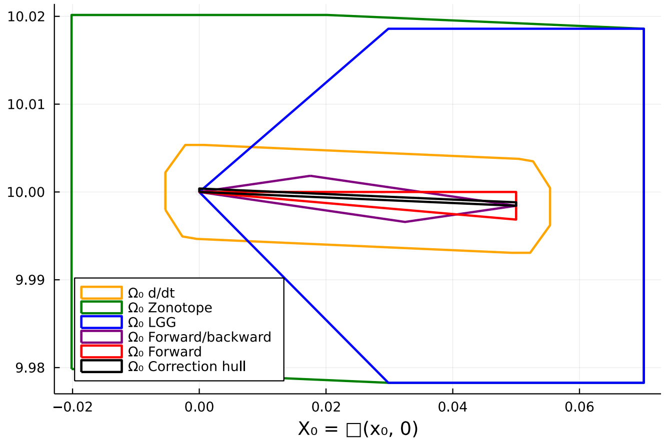


Experiment 1 - Varying δ

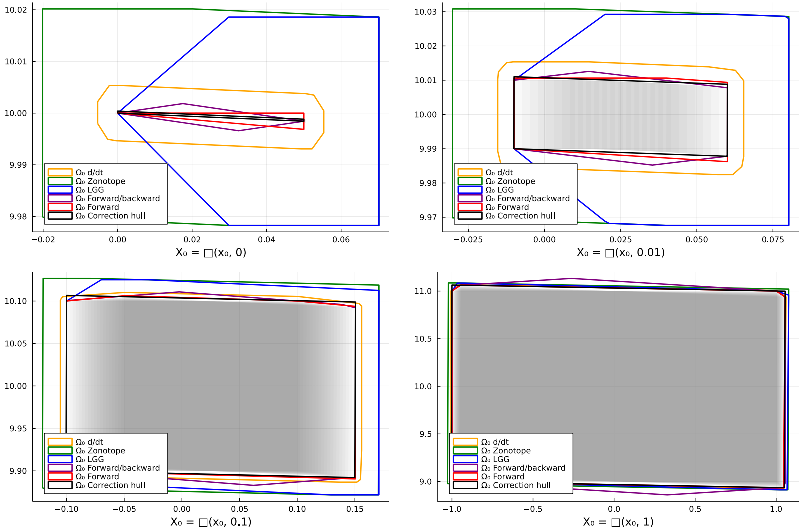


Experiment 1 - Example 2

- $\delta = 0.005$



Experiment 1 - Varying $\mathcal{X}_0$



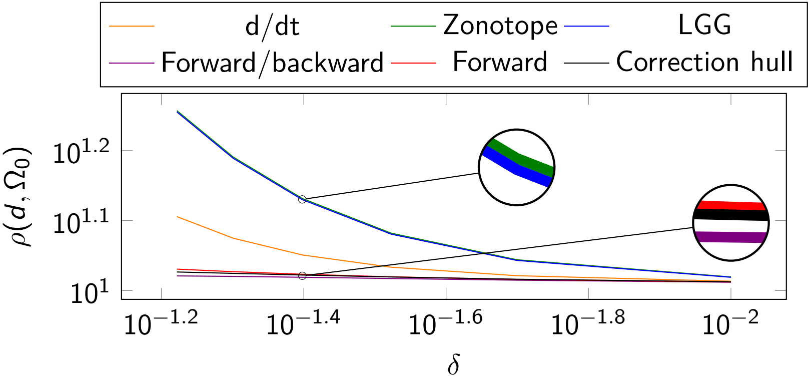
Experiment 2

- Quantitative analysis

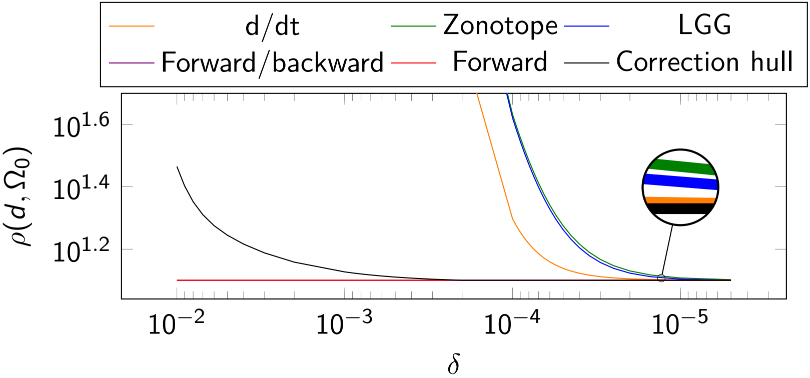
- **Harmonic oscillator**
- **Two degree of freedom**
  - 4 dimensions
  - $\|A\|_\infty = 10001$
- **ISS (docking maneuver)**
  - 270 dimensions
  - Nondeterministic inputs
  - $\|A\|_\infty = 3763$

- **Vary** δ
- Compare **support function** $\rho(d, \Omega_0)$ in **direction** $d = \mathbf{1}$
  - Lazy computation except for **Zonotope** and **Correction hull**

Experiment 2 - Harmonic oscillator

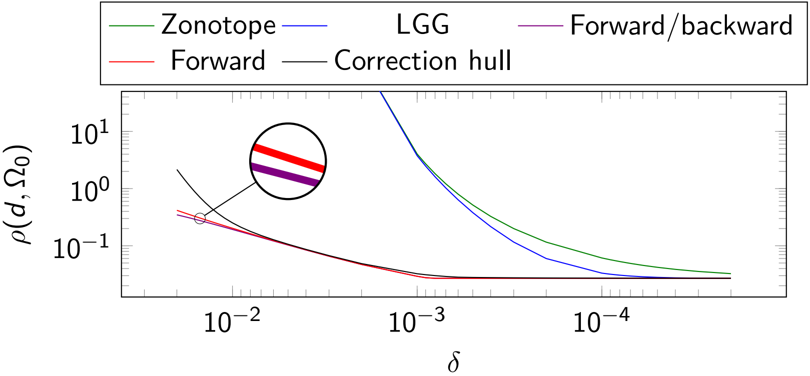


Experiment 2 - Two degree of freedom



- **Forward/backward** and **Forward** yield identical results

Experiment 2 - ISS



- **d/dt** not applicable here

Experiment 2 - Run times

- Time in milliseconds

Model	d/dt	Zonotope[1]	LGG	Fwd/bwd	Forward	Correction hull[1]
Oscillator	0.01	0.02	0.01	6.56	0.03	0.23
TDoF	0.03	0.05	0.01	6.17	0.06	0.51
ISS	–	32.99	25.93	657.80	476.96	4701.20

[1] Non-lazy computation

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Conclusion

- Six methods to **discretize** **linear continuous systems**
- Choose **time step** δ and compute $\Omega_0 \supseteq \mathcal{R}_{[0,\delta]}$
- **First-/Second-order methods**: cheap but coarse, esp. for large $\|A\|$
- **Forward-backward method**: expensive but precise
- **Forward-only method**: good compromise
- **Correction-hull method**: expensive; incomparable; yields zonotope; applies to interval matrix A
- Also in the paper:
    - **Homogenization** of systems with **inputs**
    - Two-step process with **smaller time step**
    - **Efficient implementation**
    - Computation of $e^{A\delta}$ for large A with **Krylov subspace**